Appendix A
Proof of Theorem 1

Theorem 1 states that any equilibrium point of the closed loop system under the proposed controller is also a KKT point. We proceed by showing that any equilibrium of the closed-loop microgrid yields a solution of the KKT condition.

Stationarity condition: We begin with (8c). Since the communication graph $\mathbb{G}$ is connected, it follows from (8c) that $\lambda_i = \lambda_j = \lambda$ for some constant λ for all i, j [26]. Equation (9) then immediately yields the KKT stationarity condition (5).

Primal feasibility of power balance (1b)**:** Equation (1b) corresponds to physical constraints of energy balance that is always satisfied in the system. From equation (11) of the closed-loop microgrid system, it is directly deductible (11a) that at equilibrium, $\omega_i = \omega^*$, this equality is satisfied when the power balance is achieved.

Complementary slackness of DG active power limits:

The equations (8e) and (8f) in equilibrium yield

$$0 = \mu_i^2 \cdot \max\left\{P_i + \frac{1}{\mu_i^2}k_i^5\sigma_i^+ - P_i^{\max}, 0\right\} - k_i^5\sigma_i^+ \tag{14a}$$
$$0 = \mu_i^3 \cdot \max\left\{P_i^{\min} + \frac{1}{\mu_i^3}k_i^7\sigma_i^+ - P_i, 0\right\} - k_i^7\sigma_i^- \tag{14b}$$

for all $i \in \{1, \dots, n\}$. We will prove that complementarity condition (4c) holds in equilibrium, using equation (14a). In light of (14a), the set of buses can be partitioned into two disjoint sets $U, V \subset \{1, \dots, n\}$ such that

$$P_i + \frac{1}{\mu_i^2}k_i^5\sigma_i^+ - P_i^{\max} \leq 0\,, \qquad i \in U\,, \tag{15a}$$
$$P_i + \frac{1}{\mu_i^2}k_i^5\sigma_i^+ - P_i^{\max} > 0\,, \qquad i \in V\,. \tag{15b}$$

Then equation (14a) can be reduced to

$$\begin{aligned} 0 &= -k_i^5\sigma_i^+\,, && i \in U\,,\\ 0 &= \mu_i^2(P_i - P_i^{\max})\,, && i \in V\,, \end{aligned}$$

from which we conclude that $\sigma_i^+ = 0$ for all $i \in U$ and $P_i = P_i^{\max}$ for all $i \in V$. In either case, we conclude that $\sigma_i^+(P_i - P_i^{\max}) = 0$. For $i \in U$, substitution of $\sigma_i^+ = 0$ into (15a) immediately shows that $P_i \leq P_i^{\max}$ and for $i \in V$, substitution of $P_i = P_i^{\max}$ into (15b) implies that $\sigma_i^+ > 0$. We conclude that the complementary slackness condition (4c) holds, that the upper bound in (1d) is primal feasible, and that the multipliers σ_i^+ are dual feasible. Analogous arguments using (14b) show complementary slackness for σ_i^-, primal feasibility of the lower bound in (1d), and dual feasibility of σ_i^-.

Complementarity condition of line current limits: The equation (8d) at equilibrium yields

$$\begin{aligned} 0 = &-\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell})\\ &+ \mu_i^1 \max\left\{I_\ell + \frac{1}{\mu_i^1}k_i^3\gamma_{i\ell} - I_\ell^{\max}, 0\right\} - k_i^3\gamma_{i\ell} \end{aligned} \tag{16}$$

for all $\ell \in \{1, \dots, L\}$. We will show that the complementarity condition (4b) holds. For each line ℓ, we partition the set of DGs into two disjoint subsets $R, S \subset \{1, \dots, n\}$

$$I_\ell + \frac{1}{\mu_i^1}k_i^3\gamma_{i\ell} - I_\ell^{\max} \leq 0\,, \quad i \in R\,, \tag{17a}$$
$$I_\ell + \frac{1}{\mu_i^1}k_i^3\gamma_{i\ell} - I_\ell^{\max} > 0\,, \quad i \in S\,. \tag{17b}$$

With these definitions, equation (16) reduces to

$$\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) = -k_i^3\gamma_{i\ell}\,, \quad i \in R\,, \tag{18a}$$
$$\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) = \mu_i^1(I_\ell - I_\ell^{\max})\,, \quad i \in S\,. \tag{18b}$$

The rest of the demonstration is separated in 3 cases:

- *Case I:* $S = \emptyset$ (i.e., $i \in R,\ \forall i$)
- *Case II:* $R = \emptyset$ (i.e., $i \in S,\ \forall i$)
- *Case III:* Both R and S are non-empty.

Case I: When S is empty, it follows from (18a) that

$$0 = -\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) - k_i^3\gamma_{i\ell}$$

for all $i \in \{1, \dots, n\}$. Letting $\boldsymbol{\gamma}_\ell = (\gamma_{1\ell}, \gamma_{2\ell}, \dots, \gamma_{n\ell})^\mathsf{T}$, this equation may be written in matrix form as $M\boldsymbol{\gamma}_\ell = 0$, where

$$M_{ij} = \begin{cases} -a_{ij} & \text{if} \quad i \neq j \\ \sum_{j\in\mathcal{N}(i)} a_{ij} + k_i^3 & \text{if} \quad i = j \end{cases}$$

Since $k_i^3 > 0$ for all $i \in \{1, \dots, n\}$, the symmetric matrix M has strictly positive diagonal entries and is strictly diagonally dominant; it is therefore positive definite, and we conclude that $\boldsymbol{\gamma}_\ell = 0$. We may therefore take $\gamma_\ell^* = 0$ in statement (i) of the Theorem. From (17a) then, we conclude that $I_\ell - I_\ell^{\max} \leq 0$. Therefore, the primal feasibility condition (2) is satisfied, the complementary slackness condition (4b) is satisfied, and the multipliers are dual feasible.

Case II: When the set R is empty, it follows from (18b) that

$$\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) = \mu_i^1(I_\ell - I_\ell^{\max})\,,$$

for all $i \in \{1, \dots, n\}$. Summing all these equations, we obtain

$$\sum\nolimits_{i\in\mathcal{N}}\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) = \sum\nolimits_{i\in\mathcal{N}(i)} \mu_i^1(I_\ell - I_\ell^{\max})\ \forall i, \ell$$

Since the communication graph $\mathbb{G}$ is undirected, the sum on the left is zero and we find that $\sum_{i=1}^n \mu_i^1(I_\ell - I_\ell^{max})$, which implies that $I_\ell = I_\ell^{\max}$. Substituting $I_\ell = I_\ell^{\max}$ into (17b), we find that $\gamma_{i\ell} > 0$. Substituting $I_\ell = I_\ell^{\max}$ into (18b), we find that

$$\sum\nolimits_{j\in\mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) = 0\,.$$

Since the communication graph $\mathbb{G}$ is connected, this equation holds if and only if $\gamma_{i\ell} = \gamma_{j\ell}$ for all $i, j \in \{1, \dots, n\}$.

We may therefore take $\gamma_\ell^* = \gamma_{n\ell} > 0$ in statement (i) of the theorem. We conclude that the primal feasibility condition (2) is satisfied, the complementary slackness condition (4b) is satisfied, and the multipliers γ_ℓ^* are dual feasible.

Case III: Now assume both sets S and R are non-empty. Assume first that the line ℓ is *not* congested, meaning that $I_\ell - I_\ell^{\max} \leq 0$. From (17a)–(17b) it holds that

$$(I_\ell - I_\ell^{\max}) \leq -\frac{1}{\mu_i^1} k_i^3 \gamma_{i\ell}\,, \qquad i \in R\,,$$
$$(I_\ell - I_\ell^{\max}) > -\frac{1}{\mu_i^1} k_i^3 \gamma_{i\ell}\,, \qquad i \in S\,.$$

Assuming that $k_i^3/\mu_i^1 = \kappa > 0$ for all $i \in \{1, \dots, n\}$, the above inequalities immediately imply that

$$\gamma_{i\ell} > \gamma_{j\ell}\,, \qquad i \in S\,, \quad j \in R\,. \tag{19}$$

Equation (18b) implies that

$$\sum\nolimits_{j \in \mathcal{N}(i)} a_{ij}(\gamma_{i\ell} - \gamma_{j\ell}) \leq 0\,, \quad i \in S\,.$$

Since $a_{ij} > 0$ for $j \in \mathcal{N}(i)$, this inequality implies that

$$\text{for all } i \in S \text{ there exists } j \in \mathcal{N}(i) \text{ s.t. } \gamma_{j\ell} \geq \gamma_{i\ell}\,. \tag{20}$$

Now let $\overline{\gamma}_S = \max_{i \in S} \gamma_{i\ell}$ and let $\mathcal{I}^* \subseteq S$ be the set of indices for which the maximum is achieved. We claim that there exists an $i^* \in \mathcal{I}^*$ such that i^* has a neighbour in the set R. To see that this is true, suppose that there was no such neighbour, which means that $\mathcal{N}(i^*) \subset S$ for all $i^* \in \mathcal{I}^*$. Then (20) implies that $\gamma_{j\ell} = \overline{\gamma}_S$ for all $j \in \mathcal{N}(i^*)$, and therefore that $\mathcal{N}(i^*) \subset \mathcal{I}^*$. Since the graph $\mathbb{G}$ is connected and R is non-empty, this argument can be repeated a finite number of times until we find an index $i^* \in \mathcal{I}^*$ with a neighbour in R. For this index, (20) implies that there exists a $j \in \mathcal{N}(i^*) \cap R$ such that $\gamma_{j\ell} \geq \overline{\gamma}_S$. However, this directly contradicts (19).

A similar contradiction argument can be applied in the case when the line is congested, meaning $I_\ell > I_\ell^{\max}$. We conclude that the assumption that the sets S and R are both non-empty was invalid, and therefore one set must always be empty, and we reduce to Case I or Case II.